Αξιοσημείωτες ταυτότητες

Λύσεις Ασκήσεων

Άσκηση 1

i) $(x+y)^2 = x^2 + 2xy + y^2$

ii) $(x \quad +3y)^2 = x^2 + 2 \cdot x \cdot 3y + (3y)^2 \ = x^2 + 6xy + 9y^2$

iii) $\left(\frac{x}{4} + \frac{y}{3}\right)^2 = \left(\frac{x}{4}\right)^2 + 2 \cdot \frac{x}{4} \cdot \frac{y}{3} + \left(\frac{y}{3}\right)^2 = \frac{x^2}{16} + \frac{xy}{6} + \frac{y^2}{9}$

iv) $(x^2 + y^2)^2 = (x^2)^2 + 2x^2y^2 + (y^2)^2 = x^4 + 2x^2y^2 + y^4$

v) $(2x + y)^2 = (2x)^2 + 2 \cdot 2x \cdot y + y^2 = 4x^2 + 4xy + y^2$

vi) $\left(\frac{x}{4} + y\right)^2 = \left(\frac{x}{4}\right)^2 + 2 \cdot \frac{x}{4} \cdot y + y^2 = \frac{x^2}{16} + \frac{xy}{2} + y^2$

vii) $(xy + y^2)^2 = (xy)^2 + 2xyy^2 + (y^2)^2 = x^2y^2 + 2xy^3 + y^4$

viii) $\left(x^2 + \frac{y^3}{3}\right)^2 = (x^2)^2 + 2 \cdot x^2 \cdot \frac{y^3}{3} + \left(\frac{y^3}{3}\right)^2 = x^4 + \frac{2}{3}x^2y^3 + \frac{y^6}{9}$

Άσκηση 2

i) $(x-y)^2 = x^2 - 2xy + y^2$

ii) $(6 - 4x)^2 = 36 - 2 \cdot 6 \cdot 4x + (4x)^2 = 36 - 48x + 16x^2$

iii) $\left(2x - \frac{y}{4}\right)^2 = (2x)^2 - 2 \cdot 2x \cdot \frac{y}{4} + \left(\frac{y}{4}\right)^2 = 4x^2 - x \cdot y + \frac{y^2}{16}$

iv) $\left(x^2 - \frac{\alpha\beta}{4}\right)^2 = (x^2)^2 - 2 \cdot x^2 \cdot \frac{\alpha\beta}{4} + \left(\frac{\alpha\beta}{4}\right)^2 = x^4 - \frac{x^2 \cdot \alpha\beta}{2} + \frac{\alpha^2\beta^2}{16}$

v) $(a-4)^2 = a^2 - 2 \cdot a \cdot 4 + 4^2 = a^2 - 8a + 16$

vi) $(4x - 5y)^2 = (4x)^2 - 2 \cdot 4x \cdot 5y + (5y)^2 = 16x^2 - 40xy + 25y^2$

vii) $\left(2x - \frac{1}{2x}\right)^2 = (2x)^2 - 2 \cdot 2x \cdot \frac{1}{2x} + \left(\frac{1}{2x}\right)^2 = 4x^2 - 2 + \frac{1}{4x^2}$

viii) $\left(\frac{x^2}{2} + \frac{2y^3}{3}\right)^2 = \left(\frac{x^2}{2}\right)^2 + 2 \cdot \frac{x^2}{2} \cdot \frac{2y^3}{3} + \left(\frac{2y^3}{3}\right)^2 = \frac{x^4}{4} + \frac{2}{3}x^2y^3 + \frac{4y^6}{9}$

Άσκηση 3

i) $(-x-3)^2 = [-(x+3)]^2 = (x+3)^2 = x^2 + 2 \cdot 3x + 3^2 = x^2 + 6x + 9$

ii) $(-x+4)^2 = (4-x)^2 = 4^2 - 2 \cdot 4 \cdot x + x^2 = 16 - 8x + x^2 = x^2 - 8x + 16$

iii) $(-2x + 5y)^2 = (5y - 2x)^2 = (5y)^2 - 2 \cdot 5y \cdot 2x + (2x)^2 =$
$25y^2 - 20y + 4x^2$

iv) $\left(-\frac{2x^2}{3} + \frac{5y^3}{3}\right)^2 = \left(\frac{5y^3}{3} - \frac{2x^2}{3}\right)^2 = \left(\frac{5y^3}{3}\right)^2 - 2 \cdot \frac{5y^3}{3} \cdot \frac{2x^2}{3} + \left(\frac{2x^2}{3}\right)^2 =$
$\frac{25y^6}{9} - \frac{20x^2y^3}{9} + \frac{4x^4}{9}$

Άσκηση 4

i) $(x-3)(x+3) = x^2 - 3^2 = x^2 - 9$

ii) $(1+xy)(1-xy) = 1^2 - (xy)^2 = 1 - x^2y^2$

iii) $(-a+3\beta)(a+3\beta) = (3\beta - a)(3\beta + a) = (3\beta)^2 - a^2 = 9\beta^2 - a^2$

iv) $(3xy - 4a\beta)(3xy + 4a\beta) = (3xy)^2 - (4a\beta)^2 = 9x^2y^2 - 16a^2\beta^2$

v) $(1-x^2)(1+x^2) = 1^2 - (x^2)^2 = 1 - x^4$

vi) $(x^2 - y^2)(x^2 + y^2) = (x^2)^2 - (y^2)^2 = x^4 - y^4$

vii) $(-x-2y)(-x+2y) = (-2y-x)(2y-x) = -(2y+x)(2y-x) =$
$-[(2y)^2 - x^2] = -(4y^2 - x^2) = x^2 - 4y^2$

Άσκηση 5

a) $(x+y+\omega)(x+y-\omega) = [(x+y)+\omega][(x+y)-\omega] =$
$(x+y)^2 - \omega^2 = x^2 + 2xy + y^2 - \omega^2$

b) $(a+\beta+1)(a+\beta-1) = [(a+\beta)+1][(a+\beta)-1] =$
$(a+\beta)^2 - 1^2 = a^2 + 2a\beta + \beta^2 - 1$

c) $(x-y+\omega)(x+y+\omega) = [(x+\omega)-y][(x+\omega)+y] =$
$(x+\omega)^2 - y^2 = x^2 + 2x\omega + \omega^2 - y^2$

d) $(x+y+\omega-2)(x+y-\omega+2) = [(x+y)+(\omega-2)] \cdot [(x+y)-(\omega-2)] =$
$(x+y)^2 - (\omega-2)^2 = x^2 + y^2 + 2xy - (\omega^2 - 2 \cdot \omega \cdot 2 + 2^2) =$
$x^2 + y^2 + 2xy - \omega^2 + 4\omega - 4$

Άσκηση 6

i) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

ii) $(2a+3\beta)^3 = (2a)^3 + 3 \cdot (2a)^2 \cdot 3\beta + 3 \cdot 2a \cdot (3\beta)^2 + (3\beta)^3 = 8a^3 + 4a^2 \cdot 3\beta +$
$6a \cdot 9\beta^2 + 27\beta^3 = 8a^3 + 12a^2\beta + 54a\beta^2 + 27\beta^3$

iii) $(2a-4\beta)^3 = (2a)^3 - 3 \cdot (2a)^2 \cdot 4\beta + 3 \cdot 2a + (4\beta)^2 - (4\beta)^3 =$
$8a^3 - 3 \cdot 4a^2.4\beta + 6a \cdot 16\beta^2 - 64\beta^3 = 8a^3 - 48a^2\beta + 96a\beta^2 - 64\beta^3$

iv) $(a+2)^3 = a^3 + 3a^2 \cdot 2 + 3a \cdot 2^2 + 2^3 = a^3 + 6a^2 + 12a + 8$

v) $(x-2)^3 = x^3 - 3x^2 \cdot 2 + 3x \cdot 2^2 - 2^3 = x^3 - 6x^2 + 12x - 8$

vi) $(-x-2y)^3 = [-(x+2y)]^3 = -(x+2y)^3 =$
$-[x^3 + 3x^2 \cdot 2y + 3x \cdot (2y)^2 + (2y)^3] =$

$$-(x^3+6x^2y+12xy^2+8y^3)=$$
$$-x^3-6x^2y-12xy^2-8y^3$$

vii) $(-x-2y^2)^3=[-(x+2y^2)]^3=-(x\quad+2y^2)^3=$
$-[x^3+3x^2\cdot(2y^2)+3x\cdot(2y^2)^2+(2y^2)^3]=$
$-(x^3+6x^2y^2+12xy^4+8y^6)=$
$-x^3-6x^2y^2-12xy^4-8y^6$

viii) $(-a+2)^3=(2-a)^3=2^3-3\cdot2^2\cdot a+3\cdot2\cdot a^2-a^3$
$=8-12a+6a^2-a^3$

Άσκηση 7

i) $(x+y+\omega)^2=[(x+y)+\omega]^2=$
$(x+y)^2+2(x+y)\cdot\omega+\omega^2=$
$x^2+2xy+y^2+2x\omega+2y\omega+\omega^2=$
$x^2+y^2+\omega^2+2xy+2x\omega+2y\omega$

Όμοια με το (i) βρίσκουμε:

ii) $(x-y+\omega)^2=x^2+y^2+\omega^2-2xy+2x\omega-2y\omega$

iii) $(x+y-\omega)^2=x^2+y^2+\omega^2+2xy-2x\omega-2y\omega$

Άσκηση 8

i) $-2(2+3\omega)^3-\omega(\omega+2)(\omega-2)+(\omega-1)^3=$
$-2(8+3\cdot4\cdot3\omega+3\cdot2\cdot9\omega^2+27\omega^3)-\omega(\omega^2-4)+\omega^3-3\omega^2\cdot1+3\omega\cdot1^2-1^3=$
$-2(8+36\omega+54\omega^2+27\omega^3)-\omega^3+4\omega+\omega^3-3\omega^2+3\omega-1=$
$-16-72\omega-108\omega^2-54\omega^3+4\omega-3\omega^2+3\omega-1=$
$-54\omega^3-111\omega^2-65\omega-17$

ii) $(x-y)^3-(x+y)^3+3(x-y)(x+y)=$
$x^3-3x^2y+3xy^2-y^3-(x^3+3x^2y+3xy^2+y^3)+3(x^2-y^2)=$
$x^3-3x^2y+3xy^2-y^3-x^3-3x^2y-3xy^2-y^3+3x^2-3y^2=$
$-6x^2y-2y^3+3x^2-3y^2$

Άσκηση 10

$A=\alpha^2-6\alpha\beta+2\beta^2=$
$\left(\sqrt{3}-\sqrt{2}\right)^2-6\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)+2\left(\sqrt{3}+\sqrt{2}\right)^2=$
$\left(\sqrt{3}\right)^2-2\sqrt{3}\cdot\sqrt{2}+\left(\sqrt{2}\right)^2-6\left[\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2\right]+2\left[\left(\sqrt{3}\right)^2+2\sqrt{3}\cdot\sqrt{2}+\left(\sqrt{2}\right)^2\right]=$

$$3-2\sqrt{6}+2-6(3-2)+2\left(3+2\sqrt{6}+2\right)=$$
$$5-2\sqrt{6}-6+2\left(5+2\sqrt{6}\right)=$$
$$-1-2\sqrt{6}+10+4\sqrt{6}=9+2\sqrt{6}$$

Άσκηση 11

i) $\frac{1}{\sqrt{6}-\sqrt{3}}=\frac{1\cdot(\sqrt{6}+\sqrt{3})}{(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})}=\frac{\sqrt{6}+\sqrt{3}}{(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})}=\frac{\sqrt{6}+\sqrt{3}}{(\sqrt{6})^2-(\sqrt{3})^2}=\frac{\sqrt{6}+\sqrt{3}}{6-3}=\frac{1}{3}\left(\sqrt{6}+\sqrt{3}\right)$

ii) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2}=\frac{(\sqrt{3})^2+2\sqrt{3}\cdot\sqrt{2}+(\sqrt{2})^2}{3-2}=\frac{3+2\sqrt{6}+2}{1}=5+2\sqrt{6}$